

The Jacobian of the transformation, again from Equations (9), is

$$J(u, v, w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 6.$$

We now have everything we need to apply Equation (7):

$$\begin{aligned} & \int_0^3 \int_0^4 \int_{x=y/2}^{x=(y/2)+1} \left(\frac{2x-y}{2} + \frac{z}{3} \right) dx dy dz \\ &= \int_0^1 \int_0^2 \int_0^1 (u+w) |J(u, v, w)| du dv dw \\ &= \int_0^1 \int_0^2 \int_0^1 (u+w)(6) du dv dw = 6 \int_0^1 \int_0^2 \left[\frac{u^2}{2} + uw \right]_0^1 dv dw \\ &= 6 \int_0^1 \int_0^2 \left(\frac{1}{2} + w \right) dv dw = 6 \int_0^1 \left[\frac{v}{2} + vw \right]_0^2 dw = 6 \int_0^1 (1 + 2w) dw \\ &= 6 \left[w + w^2 \right]_0^1 = 6(2) = 12. \end{aligned}$$

Exercises 15.8

Problems 1-4: Problems involving Substitutions and Transformed Regions in the Plane

a. Solve the system

$$u = x - y, \quad v = 2x + y$$

for x and y in terms of u and v . Then find the value of the Jacobian $\partial(x, y)/\partial(u, v)$.

b. Find the image under the transformation $u = x - y$, $v = 2x + y$ of the triangular region in the xy -plane with vertices $(0, 0)$, $(1, 1)$, and $(1, -2)$ in the xy -plane. Sketch the transformed region in the uv -plane.

a. Solve the system

$$u = x + 2y, \quad v = x - y$$

for x and y in terms of u and v . Then find the value of the Jacobian $\partial(x, y)/\partial(u, v)$.

b. Find the image under the transformation $u = x + 2y$, $v = x - y$ of the triangular region in the xy -plane bounded by the lines $y = 0$, $y = x$, and $x + 2y = 2$. Sketch the transformed region in the uv -plane.

a. Solve the system

$$u = 3x + 2y, \quad v = x + 4y$$

for x and y in terms of u and v . Then find the value of the Jacobian $\partial(x, y)/\partial(u, v)$.

b. Find the image under the transformation $u = 3x + 2y$, $v = x + 4y$ of the triangular region in the xy -plane bounded

by the x -axis, the y -axis, and the line $x + y = 1$. Sketch the transformed region in the uv -plane.

4. a. Solve the system

$$u = 2x - 3y, \quad v = -x + y$$

for x and y in terms of u and v . Then find the value of the Jacobian $\partial(x, y)/\partial(u, v)$.

b. Find the image under the transformation $u = 2x - 3y$, $v = -x + y$ of the parallelogram R in the xy -plane with boundaries $x = -3$, $x = 0$, $y = x$, and $y = x + 1$. Sketch the transformed region in the uv -plane.

Substitutions in Double Integrals

5. Evaluate the integral

$$\int_0^4 \int_{x=y/2}^{x=(y/2)+1} \frac{2x-y}{2} dx dy$$

from Example 1 directly by integration with respect to x and y to confirm that its value is 2.

6. Use the transformation in Exercise 1 to evaluate the integral

$$\iint_R (2x^2 - xy - y^2) dx dy$$

for the region R in the first quadrant bounded by the lines $y = -2x + 4$, $y = -2x + 7$, $y = x - 2$, and $y = x + 1$.

7. Use the transformation in Exercise 3 to evaluate the integral

$$\iint_R (3x^2 + 14xy + 8y^2) dx dy$$

for the region R in the first quadrant bounded by the lines $y = -(3/2)x + 1$, $y = -(3/2)x + 3$, $y = -(1/4)x$, and $y = -(1/4)x + 1$.

8. Use the transformation and parallelogram R in Exercise 4 to evaluate the integral

$$\iint_R 2(x - y) dx dy.$$

9. Let R be the region in the first quadrant of the xy -plane bounded by the hyperbolas $xy = 1$, $xy = 9$ and the lines $y = x$, $y = 4x$. Use the transformation $x = u/v$, $y = uv$ with $u > 0$ and $v > 0$ to rewrite

$$\iint_R \left(\sqrt{\frac{y}{x}} + \sqrt{xy} \right) dx dy$$

as an integral over an appropriate region G in the uv -plane. Then evaluate the uv -integral over G .

10. a. Find the Jacobian of the transformation $x = u$, $y = uv$ and sketch the region G : $1 \leq u \leq 2$, $1 \leq uv \leq 2$, in the uv -plane.
b. Then use Equation (2) to transform the integral

$$\int_1^2 \int_1^{2/y} \frac{y}{x} dy dx$$

into an integral over G , and evaluate both integrals.

11. **Polar moment of inertia of an elliptical plate** A thin plate of constant density covers the region bounded by the ellipse $x^2/a^2 + y^2/b^2 = 1$, $a > 0$, $b > 0$, in the xy -plane. Find the first moment of the plate about the origin. (*Hint*: Use the transformation $x = ar \cos \theta$, $y = br \sin \theta$.)
12. **The area of an ellipse** The area πab of the ellipse $x^2/a^2 + y^2/b^2 = 1$ can be found by integrating the function $f(x, y) = 1$ over the region bounded by the ellipse in the xy -plane. Evaluating the integral directly requires a trigonometric substitution. An easier way to evaluate the integral is to use the transformation $x = au$, $y = bv$ and evaluate the transformed integral over the disk G : $u^2 + v^2 \leq 1$ in the uv -plane. Find the area this way.
13. Use the transformation in Exercise 2 to evaluate the integral

$$\int_0^{2/3} \int_y^{2-2y} (x + 2y)e^{(y-x)} dx dy$$

by first writing it as an integral over a region G in the uv -plane.

14. Use the transformation $x = u + (1/2)v$, $y = v$ to evaluate the integral

$$\int_0^2 \int_{y/2}^{(y+4)/2} y^3(2x - y)e^{(2x-y)^2} dx dy$$

by first writing it as an integral over a region G in the uv -plane.

15. Use the transformation $x = u/v$, $y = uv$ to evaluate the integral sum

$$\int_1^2 \int_{1/y}^y (x^2 + y^2) dx dy + \int_2^4 \int_{y/4}^{4/y} (x^2 + y^2) dx dy.$$

16. Use the transformation $x = u^2 - v^2$, $y = 2uv$ to evaluate the integral

$$\int_0^1 \int_0^{2\sqrt{1-x}} \sqrt{x^2 + y^2} dy dx.$$

(*Hint*: Show that the image of the triangular region G with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$ in the uv -plane is the region of integration R in the xy -plane defined by the limits of integration.)

Substitutions in Triple Integrals

17. Evaluate the integral in Example 5 by integrating with respect to x , y , and z .
18. **Volume of an ellipsoid** Find the volume of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

(*Hint*: Let $x = au$, $y = bv$, and $z = cw$. Then find the volume of an appropriate region in uvw -space.)

19. Evaluate

$$\iiint |xyz| dx dy dz$$

over the solid ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1.$$

(*Hint*: Let $x = au$, $y = bv$, and $z = cw$. Then integrate over an appropriate region in uvw -space.)

20. Let D be the region in xyz -space defined by the inequalities

$$1 \leq x \leq 2, \quad 0 \leq xy \leq 2, \quad 0 \leq z \leq 1.$$

Evaluate

$$\iiint_D (x^2y + 3xyz) dx dy dz$$

by applying the transformation

$$u = x, \quad v = xy, \quad w = 3z$$

and integrating over an appropriate region G in uvw -space.

Theory and Examples

21. Find the Jacobian $\partial(x, y)/\partial(u, v)$ of the transformation
a. $x = u \cos v$, $y = u \sin v$
b. $x = u \sin v$, $y = u \cos v$.
22. Find the Jacobian $\partial(x, y, z)/\partial(u, v, w)$ of the transformation
a. $x = u \cos v$, $y = u \sin v$, $z = w$
b. $x = 2u - 1$, $y = 3v - 4$, $z = (1/2)(w - 4)$.

23. Evaluate the appropriate determinant to show that the Jacobian of the transformation from Cartesian $\rho\phi\theta$ -space to Cartesian xyz -space is $\rho^2 \sin \phi$.
24. **Substitutions in single integrals** How can substitutions in single definite integrals be viewed as transformations of regions? What is the Jacobian in such a case? Illustrate with an example.
25. **Centroid of a solid semiellipsoid** Assuming the result that the centroid of a solid hemisphere lies on the axis of symmetry three-eighths of the way from the base toward the top, show, by transforming the appropriate integrals, that the center of mass of a solid semiellipsoid $(x^2/a^2) + (y^2/b^2) + (z^2/c^2) \leq 1$, $z \geq 0$, lies on the z -axis three-eighths of the way from the base toward the top. (You can do this without evaluating any of the integrals.)
26. **Cylindrical shells** In Section 6.2, we learned how to find the volume of a solid of revolution using the shell method; namely, if the region between the curve $y = f(x)$ and the x -axis from a to b ($0 < a < b$) is revolved about the y -axis, the volume of the resulting solid is $\int_a^b 2\pi x f(x) dx$. Prove that finding volumes by

using triple integrals gives the same result. (*Hint:* Use cylindrical coordinates with the roles of y and z changed.)

27. **Inverse transform** The equations $x = g(u, v)$, $y = h(u, v)$ in Figure 15.54 transform the region G in the uv -plane into the region R in the xy -plane. Since the substitution transformation is one-to-one with continuous first partial derivatives, it has an inverse transformation and there are equations $u = \alpha(x, y)$, $v = \beta(x, y)$ with continuous first partial derivatives transforming R back into G . Moreover, the Jacobian determinants of the transformations are related reciprocally by

$$\frac{\partial(x, y)}{\partial(u, v)} = \left(\frac{\partial(u, v)}{\partial(x, y)} \right)^{-1} \quad (10)$$

Equation (10) is proved in advanced calculus. Use it to find the area of the region R in the first quadrant of the xy -plane bounded by the lines $y = 2x$, $2y = x$, and the curves $xy = 2$, $2xy = 1$ for $u = xy$ and $v = y/x$.

28. (*Continuation of Exercise 27.*) For the region R described in Exercise 27, evaluate the integral $\iint_R y^2 dA$.

Chapter 15 Questions to Guide Your Review

- Define the double integral of a function of two variables over a bounded region in the coordinate plane.
- How are double integrals evaluated as iterated integrals? Does the order of integration matter? How are the limits of integration determined? Give examples.
- How are double integrals used to calculate areas and average values. Give examples.
- How can you change a double integral in rectangular coordinates into a double integral in polar coordinates? Why might it be worthwhile to do so? Give an example.
- Define the triple integral of a function $f(x, y, z)$ over a bounded region in space.
- How are triple integrals in rectangular coordinates evaluated? How are the limits of integration determined? Give an example.
- How are double and triple integrals in rectangular coordinates used to calculate volumes, average values, masses, moments, and centers of mass? Give examples.
- How are triple integrals defined in cylindrical and spherical coordinates? Why might one prefer working in one of these coordinate systems to working in rectangular coordinates?
- How are triple integrals in cylindrical and spherical coordinates evaluated? How are the limits of integration found? Give examples.
- How are substitutions in double integrals pictured as transformations of two-dimensional regions? Give a sample calculation.
- How are substitutions in triple integrals pictured as transformations of three-dimensional regions? Give a sample calculation.

Chapter 15 Practice Exercises

Evaluating Double Iterated Integrals

In Exercises 1–4, sketch the region of integration and evaluate the double integral.

1. $\int_1^{10} \int_0^{1/y} ye^{xy} dx dy$

2. $\int_0^1 \int_0^{x^3} e^{y/x} dy dx$

3. $\int_0^{3/2} \int_{-\sqrt{9-4t^2}}^{\sqrt{9-4t^2}} t ds dt$

4. $\int_0^1 \int_{\sqrt{y}}^{2-\sqrt{y}} xy dx dy$

In Exercises 5–8, sketch the region of integration and write an equivalent integral with the order of integration reversed. Then evaluate both integrals.

5. $\int_0^4 \int_{-\sqrt{4-y}}^{(y-4)/2} dx dy$

6. $\int_0^1 \int_{x^2}^x \sqrt{x} dy dx$

7. $\int_0^{3/2} \int_{-\sqrt{9-4y^2}}^{\sqrt{9-4y^2}} y dx dy$

8. $\int_0^2 \int_0^{4-x^2} 2x dy dx$